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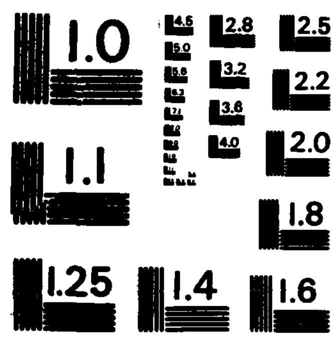

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Richard E. Barlow and Xiang Zhang*

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ADA 166189

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC 86-2	2. GOVT ACCESSION NO. N/A	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) Bayesian Analysis of Inspection Sampling Procedures Discussed by Deming		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL Report
		6. PERFORMING ORG. REPORT NUMBER ORC 86-2
7. AUTHOR(s) Richard E. Barlow and Xiang Zhang		8. CONTRACT OR GRANT NUMBER(s) DAAG29-85-K-0208
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center 3115 Etcheverry Hall University of California, Berkeley Berkeley, CA 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE January 1986
		13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inspection Sampling
Bayes
Influence Diagrams
Quality Assurance
Decision Analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

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BAYESIAN ANALYSIS OF INSPECTION SAMPLING PROCEDURES DISCUSSED BY DEMING[1]

by

Richard E. Barlow and Xiang Zhang

1. ABSTRACT

In Chapter 13 of Quality, Productivity, and Competitive Position, Deming discusses in detail inspection sampling relative to two different cost setups. Because of their practical importance, one of these models was examined in detail from a Bayesian point of view by Barlow and Zhang (1985). A very general Bayesian treatment of these problems is provided by Lorenzen (1985). A computer program is also provided by Lorenzen for the Bernoulli case as well as inspection sampling plans which reject the lot when the number of defective items is sufficiently large. However, as we shall see, simple rules of this form are not optimal for all of Deming's models. In particular, Deming's discussion of inspection rules when the finished assembly cannot be repaired by replacing defective units, is of this type. Deming describes this model as "value added to substrate". For example, if the unit is a bag

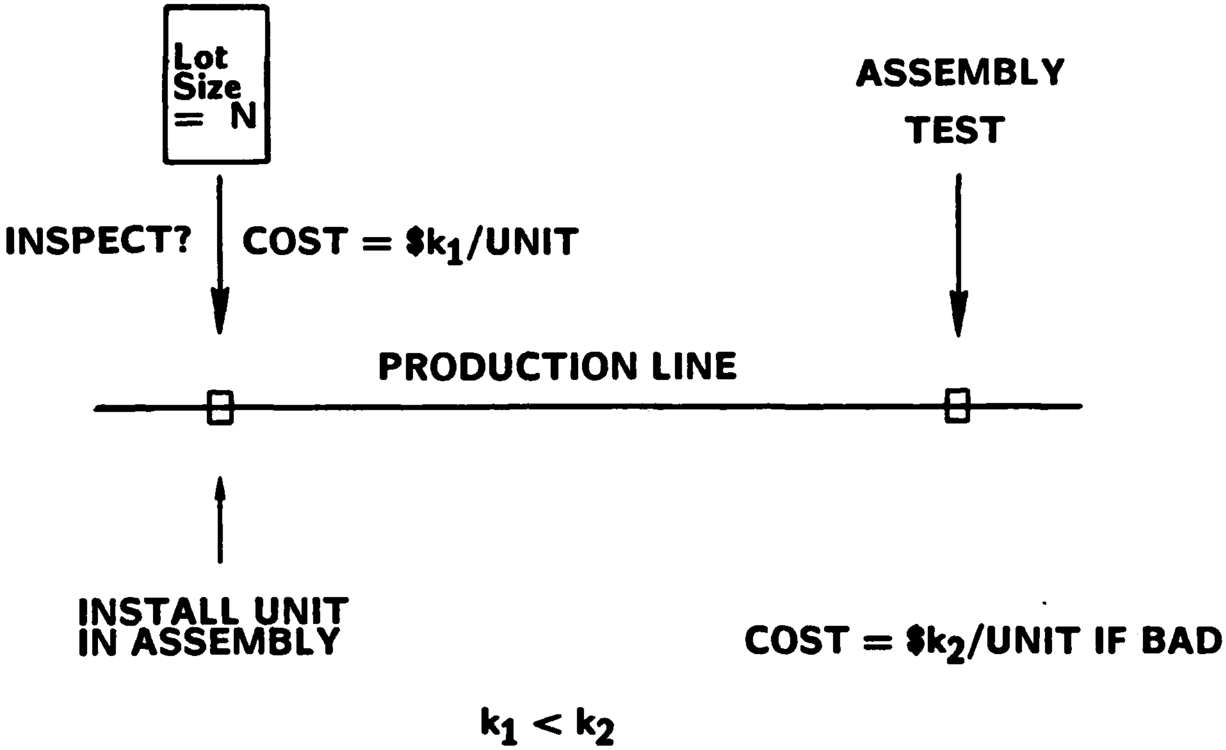
[1]Research supported by the U.S. Army Research Office, under contract DAAG29-85-K-0208 with the University of California.

of cement powder, then it cannot be recovered from the assembly after water has been added.

The purpose of this paper is to analyze and contrast the two models considered by Deming from a Bayesian decision analysis point of view. We will analyze the models using influence diagrams, [Howard and Matheson (1981), Shachter (1984)]. This is a relatively new technique for studying statistical problems. It will be useful for studying inspection sampling problems.

2. USING INFLUENCE DIAGRAMS

We will explain the use of influence diagrams in terms of a very simple model. Suppose we are considering whether or not to inspect units in a unique lot of size N of such units. That is, there are no other units than those in this lot. The question is whether or not to inspect units before installing them in an assembly. Inspection costs k_1 dollars per unit. If a unit is not inspected but is found later to be defective in the assembly, the cost is k_2 dollars. The decision variables are n, the inspection sample size, and the inspection decision after all n units have been inspected. Inspection decision d_0 is not to continue inspection. Inspection decision d_1 is to inspect the remaining $N-n$. Figure 2.1 describes the problem setup.



DEMING'S INSPECTION PROBLEM

FIGURE 2.1

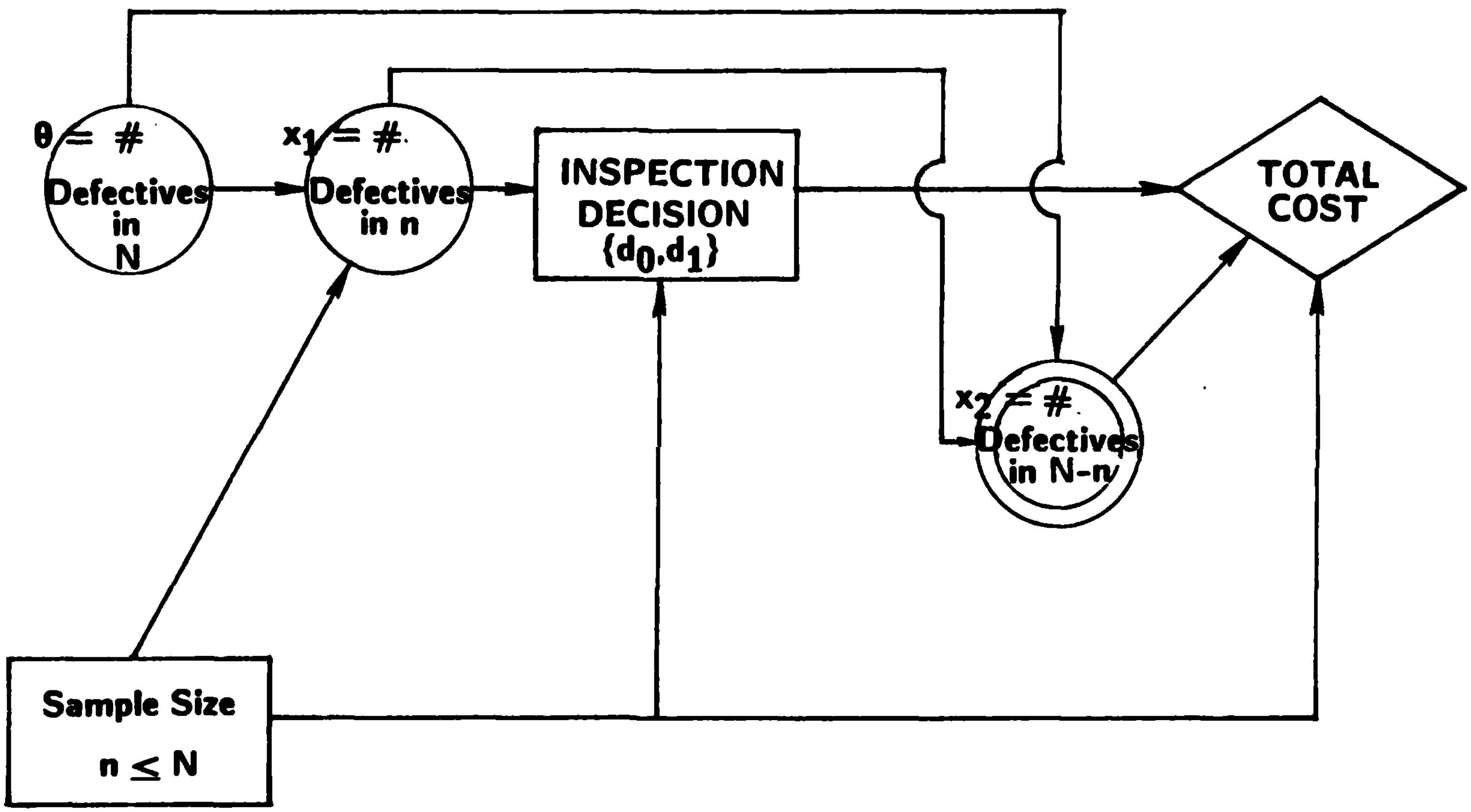
Figure 2.2 is an influence diagram related to our problem. In the diagram, circles denote unknown quantities while rectangles denote decision nodes. A double circle denotes a deterministic or logical node. In the figure, the node corresponding to x_2, the number of defectives in the remainder N-n, is deterministic since

$$x_2 = \theta - x_1 \quad \text{given} \quad \theta \quad \text{and} \quad x_1 .$$

Arrows indicate logical and statistical dependence. Associated with each circle is a conditional probability distribution. If units in the lot are judged exchangeable a priori, then the distribution of x_1 given θ, n, and N is hypergeometric. An initial or prior distribution must be assigned to θ, the unknown number of defectives in the lot. Given n and x_2 the total cost is $k_1 n + k_2 x_2$ if the decision is d_0. If the decision is d_1, the total cost is $k_1 N$. [redacted] See Barlow and Zhang (1985) for a related model.

A mathematically convenient distribution model for the initial distribution of θ is the Beta-Binomial. θ is Beta-Binomial if θ given p is Binomial(p,N) and p is Beta(A,B) where

$$\text{Beta}(A,B) \approx p^{A-1}(1-p)^{B-1}$$



INSPECTION OF A UNIQUE LOT

FIGURE 2.2

and $A, B \geq 0$. In this case [cf. Basu and Pereira (1982)]

$$E[\theta - x_1 | x_1, n, N] = (N-n)(A+x_1)/[A+B+n]$$

and the decision rule d_1 is optimal if

$$(A+x_1)/[A+B+n] \geq k_1/k_2 \ .$$

Since the left hand side is increasing in x_1 and decreasing in $n \leq N$, we have that $n=N$ is optimal if and only if

$$A/[A+B+N-1] \geq k_1/k_2 \ .$$

Unfortunately the Beta-Binomial is not an appropriate prior distribution if we strongly believe a priori that θ is approximately θ_o $(0 < \theta_o < N)$ so that the coefficient of variation

$$\sigma(\theta)/\mu(\theta) < [(1-\theta_o/N)/\theta_o]^{1/2} \ .$$

For the Beta-Binomial

$$\sigma(\theta) = \sqrt{[NA/(A+B)][1-A/(A+B)][(A+B+N)/(A+B+1)]}$$

and $\mu(\theta) = NA/(A+B) = \theta_o$ so that the coefficient of variation satisfies

$$\sigma(\theta)/\mu(\theta) = \sqrt{\frac{(1-\theta_o/N)}{\theta_o}\frac{(A+B+N)}{(A+B+1)}}$$

$$> \sqrt{(1-\theta_o/N)/\theta_o} \ .$$

In this case d_o may be best if x_1 is sufficiently large so that the decision rule is reversed from the

Beta-Binomial case.

In the appendix the solution strategy for the problem with arbitrary prior is described in terms of arc reversals and node eliminations. To determine the sample size, n ,

$$\underset{n}{\text{Minimize}} \sum_{x_1} \text{Min}\{k_1 n + k_2 E(x_2 \mid x_1, n, N),\ k_1 N\}\ p(x_1 \mid n, N)\ .$$

Knowing the decision rule (as in the Beta-Binomial case) simplifies the computation by allowing the first minimum computation before the summation.

3. DEMING'S INSPECTION SAMPLING MODELS

Units from a lot of size N are installed in an assembly at a point in the production line. The question is whether or not to inspect the unit (at cost $\$k_1$/unit) before installing the unit in an assembly. If the unit is not inspected but is defective, then this will be discovered at assembly test. A cost $\$k_2 > \$k_1$ will then be incurred. In model 1, the defective unit found at assembly test will be replaced by a good unit. In model 2, this is not possible and the assembly will be sold at a reduced price. In this case k_2 is the loss incurred. In Deming's models all lots are judged exchangeable a priori so that a binomial(p,n) probability function is used for the number of defective units in the lot. The unknown percent defective is denoted by p.

In both models we consider inspecting $n \leq N$ units. Let x_1 be the number defective in the sample of size n. Since known defective units will not be installed, an additional y_1 units are inspected until good units are found and installed. Since the vendor will replace defective units at the vendor's cost we may consider the y_1 units that are inspected to come from a separate and very large lot provided by the vendor. The sampling inspection cost is $k_1[n+y_1]$.

After inspecting n units we consider only two possible decisions for both models, namely

d_0: STOP inspecting units

d_1: INSPECT the remaining $N-n$.

In model 1, the x_2 defective units found in the remaining $N-n$ units at assembly test are replaced by good units. To do this an additional y_2 units need to be inspected to find good units. The y_2 units inspected also come from the separate lot provided by the vendor. The cost of defective units found at assembly test is

$$k_2x_2 + k_1y_2 \ .$$

The logical and statistical dependency between the unknown quantities and decisions are shown in Figure 3.1. The random quantity, p , is the unknown percent defective in many lots, all judged exchangeable.

Figure 3.1 makes explicit the well known fact that x_1 and x_2 given n and p are statistically independent. Table 3.1 describes the variables used in Figure 3.1.

TABLE 3.1

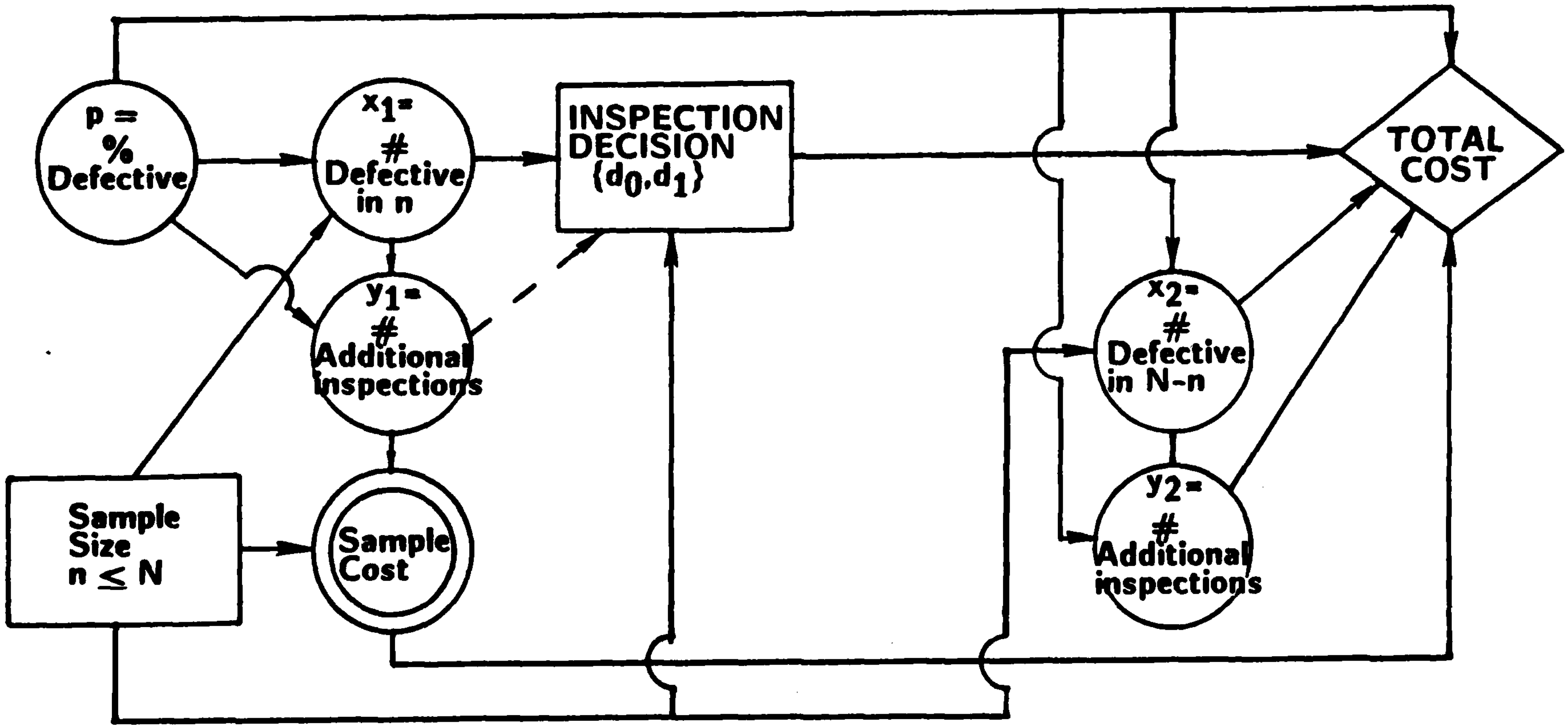
MODEL 1

UNKNOWN QUANTITIES	DESCRIPTION	CONDITIONAL PROB. DISTRIBUTIONS
p	Percent defective in many lots	$\pi(p)$
x_1	# defective in sample size n	$p(x_1 \mid n,p)$
y_1 $(y_1 \geq x_1)$	# additional inspections rqd.	$p(y_1 \mid x_1,p)$
x_2	# defectives in N-n	$p(x_2 \mid N-n,p)$
y_2 $(y_2 \geq x_2)$	# additional inspections rqd.	$p(y_2 \mid x_2,p)$

DECISION VARIBLES	DESCRIPTION
n	Sample Size
$d_0(x_1,n)$	Decision to STOP inspection given (x_1,n)
$d_1(x_1,n)$	Decision to INSPECT remaining N-n given (x_1,n)

d_0: $k_1(n+y_1) + k_2x_2 + k_1y_2$

d_1: $k_1[N+y_1+y_2]$



MODEL 1

FIGURE 3.1

Model 1 was examined in detail in Barlow and Zhang (1985). Let Z_i be an indicator for the i-th unit which is 1 if the unit is defective and 0 otherwise. Given n and x_i, d_i is optimal if $E[E(Z_i|p)|x_i,n] \geq k_1/k_2$. The assumption made was that only x_i and n were known at the time of the inspection decision. Since y_i, the additional inspections required is also presumably known, there is a loss of information. If both n and y_i are known at the time of the inspection decision then the critical number, c(n), may exceed n in this case. Other than this, all the results in Barlow and Zhang (1985) carry over to the case that y_i is known at the time of the inspection decision.

4. DEMING'S MODEL FOR VALUE ADDED TO SUBSTRATE

Work is done on incoming material, the substrate. The finished product will be classed as first grade or second grade or third grade or scrap. Let k_2 be the average loss from downgrading the final product or for scraping finished items.

If the incoming unit is non-defective, then it will result in a final product of first grade; otherwise, if it is defective, and if it goes into the production line, then a final product of downgrade or scrap will be produced. Every final product is subject to assembly grading. Figure 4.1 is an influence diagram representation of our problem. In this case there is an arrow from inspection decision to y_2 since $y_2 = 0$ if decision d_o is taken.

The optimal inspection decision rule can be calculated for every sample size n in terms of the costs k_1, k_2 and the prior distribution for p.

<u>Lemma 1</u>. Suppose n units are inspected and x_1 are found defective. The optimal inspection decision rule in this case is d_1 if

$$E[Var(Z_1|p) \mid x_1, n-1] \geq k_1/k_2$$

where expectation is with respect to the posterior distribution of p given x_1 and $n-1$.

Since $p(1-p) \leq 1/4$ for all $0 \leq p \leq 1$, it follows from Lemma 1 that if $k_1/k_2 > 1/4$, then d_o is optimal for all possible $0 \leq x_1 \leq n \leq N$; i.e. $n = 0$ is optimal in this case.

For a Beta(A,B) prior and given x_1 and n , d_1 is optimal if

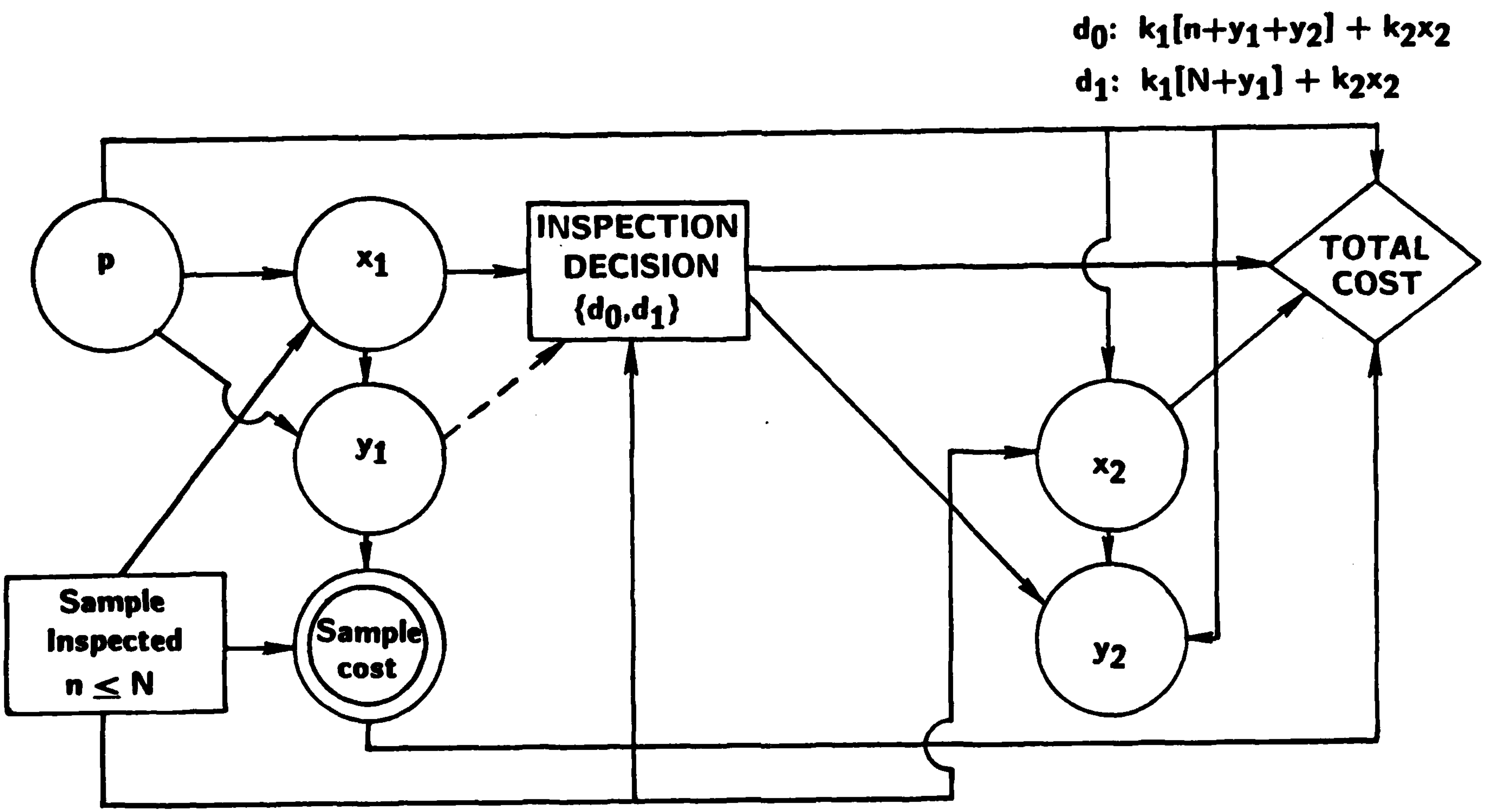
$$(A+x_1)(B+n-1-x_1)/[(A+B+n)(A+B+n-1)] \geq k_1/k_2 .$$

Hence, given x_1 and n , d_1 is optimal if

$$x_1 \in [c_1(n) , c_2(n)]$$

where $c_1(n)$ and $c_2(n)$ are given by

$$c_1(n) = \left[\frac{-b - \sqrt{(b^2 - 4ac)}}{2a}\right]$$



MODEL 2

FIGURE 4.1

$$c_2(n) = \left[\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}\right]$$

where $a = 1$, $b = -(B+n-1-A)$,

$$c = -A(B+n-1) + (A+B+n)(A+B+n-1)k_1/k_2 .$$

The intuitive reason for this rule is as follows. If the expected variance is small, then we expect that p is either close to zero or close to one. If we believe p is close to 0 so that there will be few defectives, then there is no point to inspection. On the other hand, if we believe p is close to one, then there will be so many defective units that the cost of inspecting units to find a good unit to replace a defective unit before assembly will tend to exceed the cost due to downgrading the final assembly even if it is defective.

This decision rule is in contrast to the decision rule for Model 1 where we inspect all units if the posterior expected probability that a unit is defective is sufficiently large.

Proof of Lemma 1. Suppose we inspect n units and x_1 are found defective. The expected cost given n and x_1 if we take decision d_0 is

$$c(d_0) = (N-n) \int_0^1 k_2 p \pi(p|x_1,n)\, dp .$$

If decision d_1 is taken, then the expected cost is

$$C(d_1) = (N-n) \int_0^1 k_1 (1-p)^{-1} \pi(p|x_1,n)\, dp .$$

Consider the difference

$$C(d_o) - C(d_1) = (N-n)k_2 \int_0^1 [p - k_1/k_2(1-p)^{-1}]\pi(p|x_1,n)\,dp$$

$$= (N-n)k_2 \int_0^1 (1-p)^{-1}[p(1-p) - k_1/k_2]\pi(p|x_1,n)\,dp \;.$$

Since $\pi(p|x_1,n)$ is proportional to $p^{x_1}(1-p)^{n-x_1}\pi(p)$

$$C(d_o) < C(d_1) \text{ iff } \int_0^1 [p(1-p) - k_1/k_2]p^{x_1}(1-p)^{n-1-x_1}\pi(p)\,dp < 0$$

$$\text{iff } \int_0^1 p(1-p)\pi(p|x_1,n-1)\,dp < k_1/k_2$$

$$\text{iff } \int_0^1 \mathrm{Var}(Z_1|p)\pi(p|x_1,n-1)\,dp < k_1/k_2$$

$$\text{iff } E[\mathrm{Var}(Z_1|p)|x_1,n-1] < k_1/k_2 \;. \qquad \text{QED}$$

Lemma 2. $E[\mathrm{Var}(Z_1|p)|x_1,n-1]$ is unimodal in x_1 (n) when n (x_1) is fixed.

Proof. $E[\mathrm{Var}(Z_1|p)|x_1,n-1] = \int_0^1 p(1-p)\pi(p|x_1,n-1)\,dp$.

Note that $\pi(p|x_1,n-1)$ can be written as $g(x_1,n)(1-p)p^{x_1}(1-p)^{n-x_1}$. It is well known that $p^{x_1}(1-p)^{n-x_1}$ is a totally positive function of x_1 and p and reverse regular function of n and p [Karlin Theorem 2.1, page 18 (1968)]. It follows that

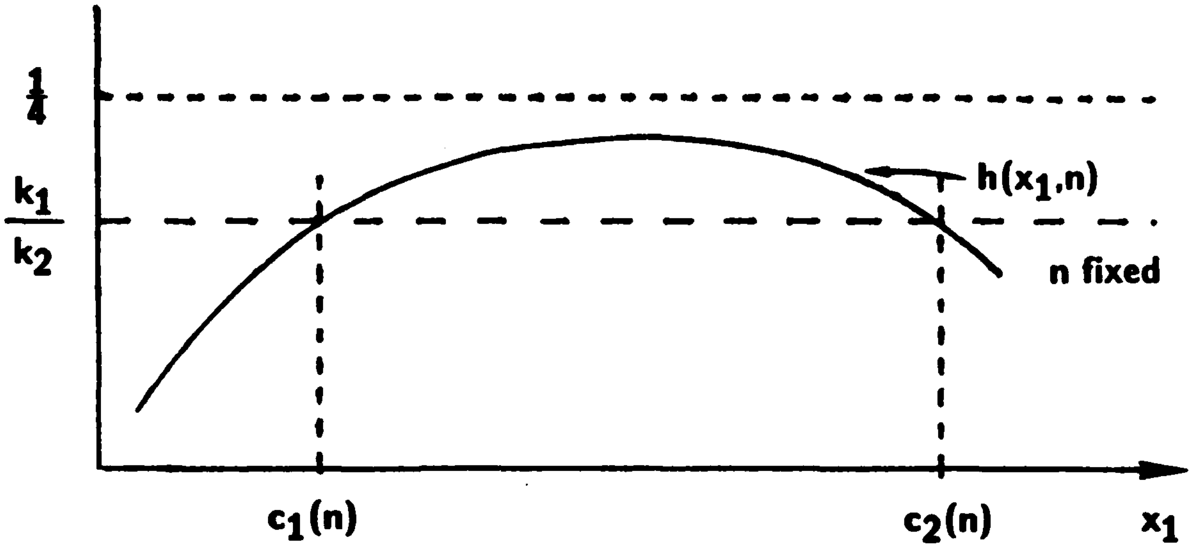
$\pi(p|x_1,n-1)$ is totally positive in x_1 and p and reverse regular in n and p since $g(x_1,n) > 0$. Using the sign variation diminishing properties of sign-regular functions [Section 3, Chapter 5, Karlin (1968)] we can finish the proof. To do this consider

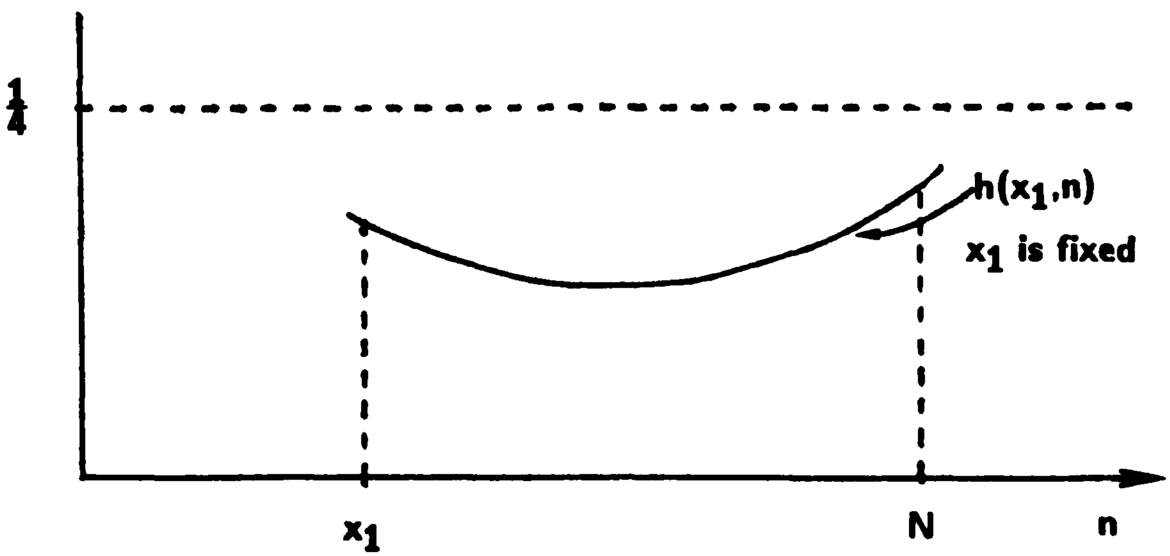
$$h(x_1,n) - a = \int_0^1 [p(1-p) - a]\pi(p|x_1,n-1)\,dp$$

$$= \int_0^1 [-p^2+p-a]\pi(p|x_1,n-1)\,dp$$

for a fixed real a. The quadratic function can have at most two sign changes in $[0,1]$. The sign must change from negative to positive to negative if there are exactly two sign changes. It follows that $h(x_1,n) - a$ as a function of x_1 (n) when n (x_1) is fixed, changes sign at most two times when x_1 (n) runs from 0 (x_1) to n (N). Furthermore $h(x_1,n)$ as a function of x_1 changes sign in the same order if there are two sign changes. As a function of n it changes sign in the opposite order if there are two sign changes. Since a is an arbitrary real number, this implies the unimodality properties as stated. QED

The figures below illustrate $h(x_1,n)$ as a function of x_1 for fixed n and also as a function of n for fixed x_1.


$\frac{1}{4}$
$\frac{k_1}{k_2}$
$h(x_1,n)$
n fixed
$c_1(n)$
$c_2(n)$
x_1


$\frac{1}{4}$
$h(x_1,n)$
x_1 is fixed
x_1
N
n

Theorem. For Model 2 and given n and x_1 the optimal inspection decision is d_1 if and only if

$$x_1 \in [c_1(n), c_2(n)] .$$

where $c_1(n)$ and $c_2(n)$ are solutions to

$$\int_0^1 p(1-p)\pi(p|c,n-1)\, dp = 0 .$$

This is true for any prior distribution for p.

The proof is an immediate consequence of Lemmas 1 and 2.

5. CONCLUSIONS

Influence diagrams are a useful technique for displaying the logical and statistical dependence between unknown quantities and decision variables in inspection sampling problems. They can also be used to develop a computational solution strategy. The solution strategy is simplified if the form of the optimal inspection decision rule can be calculated. However, the "obvious" inspection rule of inspecting all remainders in a lot when the number of defectives in a sample is sufficiently large is not always correct. This has been shown for two cases; namely the case of a unique lot when we have a strong prior for θ (i.e. the prior coefficient of variation is sufficiently small) and for the Deming Model 2 corresponding to "value added to substrate."

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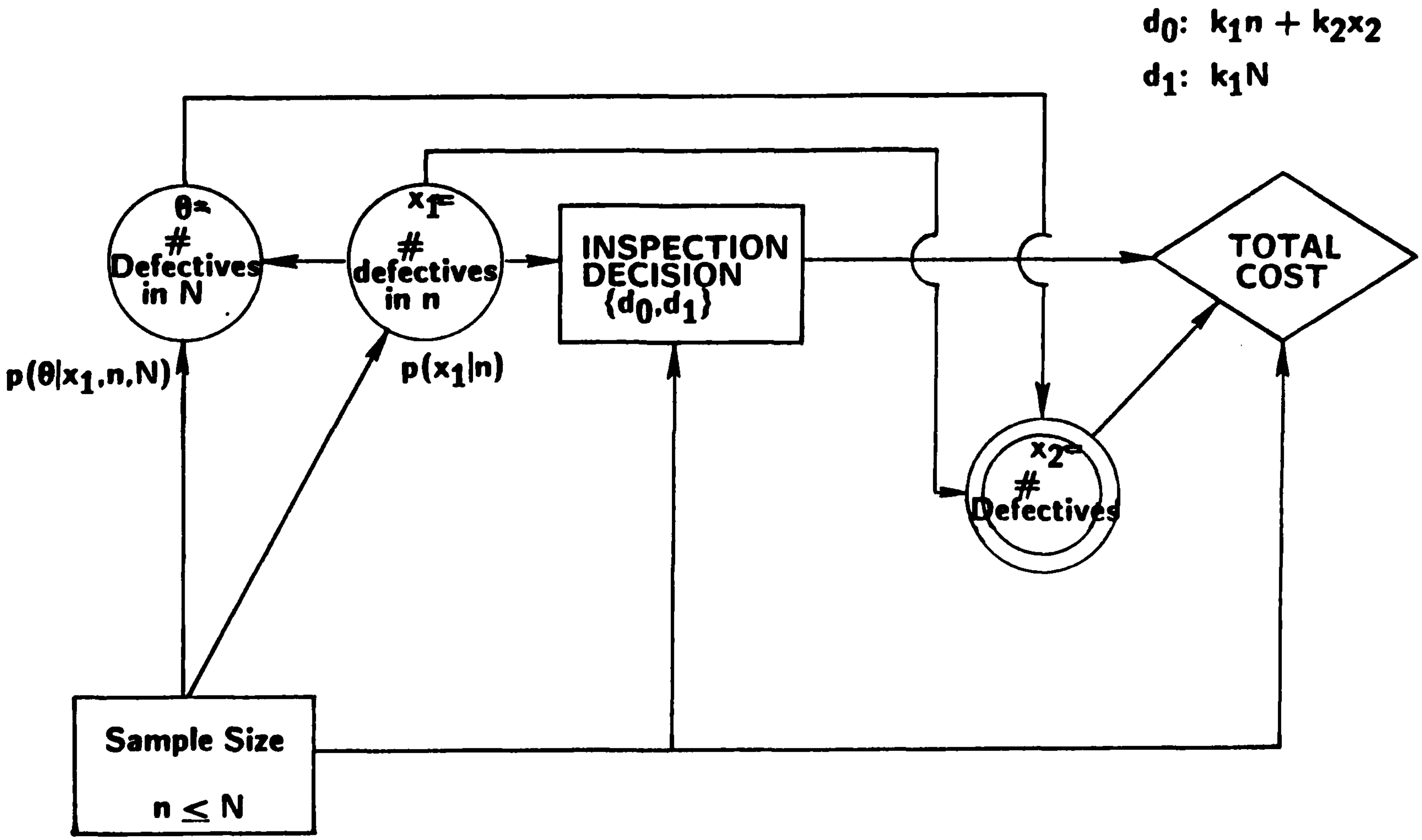
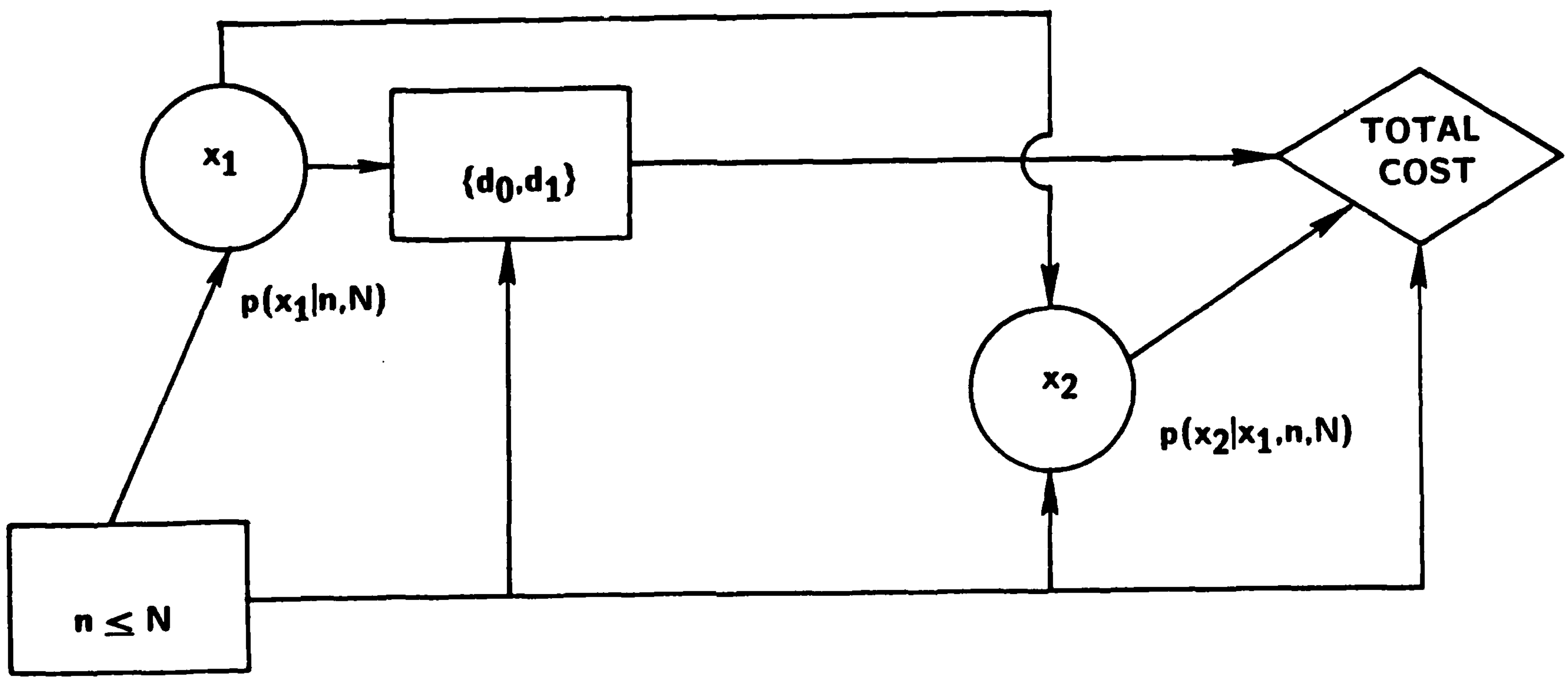
APPENDIX



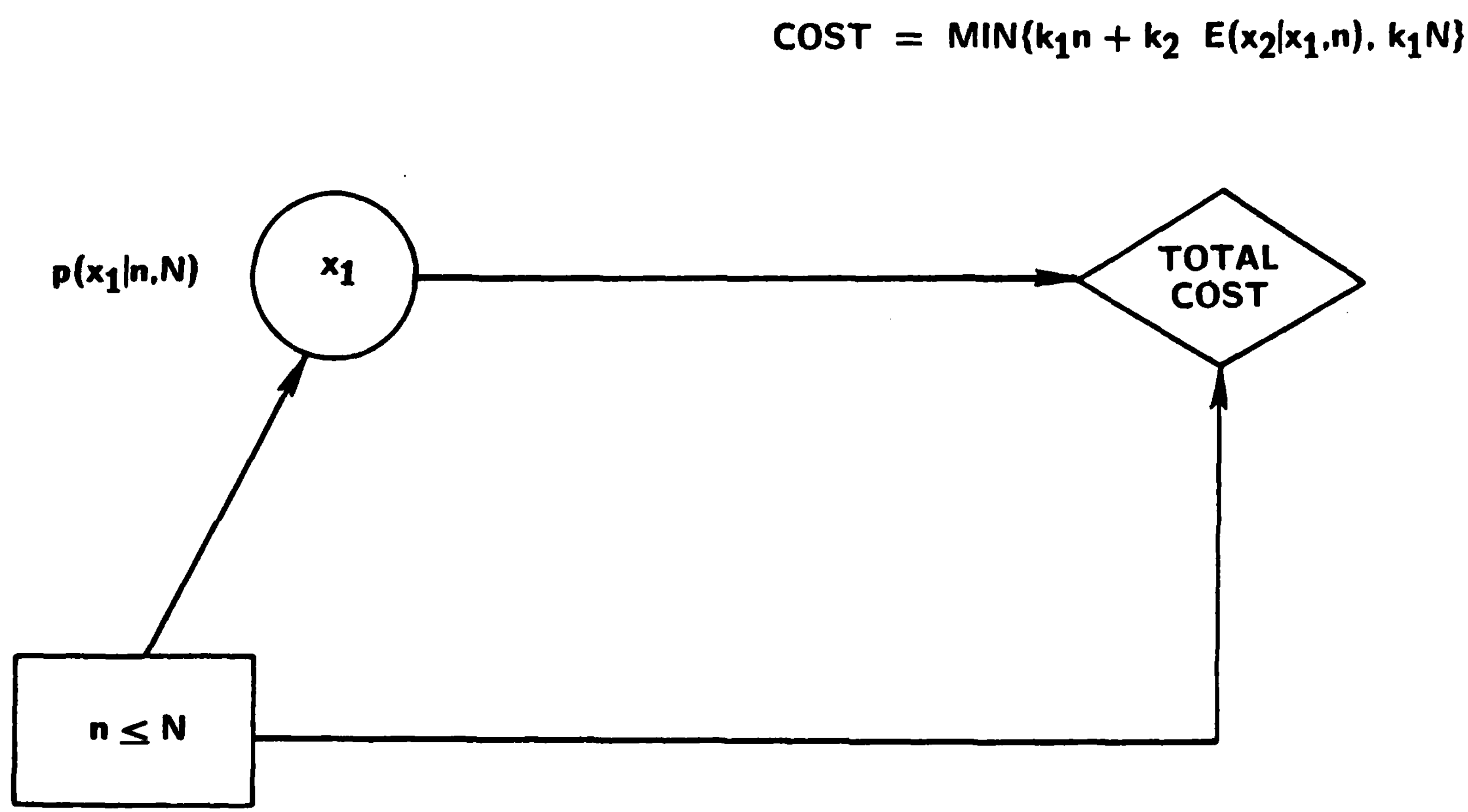
BAYES THEOREM USED TO REVERSE ARC FROM θ to x_1

FIGURE A.1



THE NODE CORRESPONDING TO θ HAS BEEN ELIMINATED

FIGURE A.2



COST HAS BEEN EXPECTED WITH RESPECT TO X_2
AND OPTIMUM DECISION RULE CHOSEN

FIGURE A.3

Knowing form of inspection decision rule $\displaystyle\sum_{x_1 \le [A+B+n]\frac{k_1}{k_2} - A} \{k_1 n + k_2 E(x_2 | x_1, n)\}\, p(x_1 | n, N) + \sum_{x_1 \ge [A+B+n]\frac{k_1}{k_2} - A} k_1 N\, p(x_1 | n, N)$

$$\text{COST} = \sum_{x_1} \text{Min}\{k_1 n + k_2 E(x_2 | x_1, n),\ k_1 N\}\ p(x_1 | n, N)$$

COST expected out with respect to distribution of x_1

The final step is to Minimize Expected Total Cost with respect to $n \le N$.

FIGURE A.4

END

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